Chapter 5

Principles of Unsteady - State Heat Transfer

In this chapter, we will study chemical processes where heat transfer is taking place due to a temperature difference within the system which is changing with time. The following problem modules illustrate different examples where unsteady – state heat transfer processes are occurring in fuel cell vehicles and in the processes for producing fuel for fuel cells.

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Example 5.2-1: Cooling of a Cylindrical Solid – Oxide Fuel Cell

A cylindrical solid – oxide fuel cell has an inner radius of 3.1 mm, outer radius of 3.9 mm and a length of 0.2 m. The fuel cell initially at a uniform temperature of 873.15 K enters a medium whose temperature is 60°C. Determine how much time in minutes is required for the fuel cell to be cooled down to a temperature of 340.23 K if the convective coefficient of the medium is $12\frac{\text{W}}{\text{m}^2\cdot\text{K}}$.

The properties of the membrane electrode assembly for this type of fuel cell are estimated by Xue et al. [1] to be:

$$\rho = 6337.3\frac{\text{kg}}{\text{m}^3} \qquad C_p = 594.3\frac{\text{J}}{\text{kg}\cdot\text{K}} \qquad k = 2.53\frac{\text{W}}{\text{m}\cdot\text{K}}$$

Strategy

This problem can be solved using the simplified equations for systems with negligible internal resistance.

Solution

The equation for the dimensionless temperature as a function of time is given by:

$$\frac{T - T_\infty}{T_0 - T_\infty} = e^{-\left(\frac{h}{C_p \rho x_1}\right)t}$$

In this equation:

T_∞ = Temperature of the medium

T_0 = Initial temperature of the fuel cell

x_1 = Characteristic dimension of the body

However, this equation is only applicable when the Biot number N_{Bi} is less than 0.1. Hence, we need to obtain this dimensionless number first, defined as:

N_{Bi} = __________

For a cylindrical object, the characteristic length is calculated as follows:

$$x_1 = \frac{r}{2}$$

1. Xue, X., Tang, J., Sammes, N., Du, Y., *Journal of Power Sources*, 142, 211−222 (2005)

In this case, since the fuel cell is a hollow cylinder, the characteristic dimension will be obtained as shown below:

$$x_1 = \frac{r_o - r_i}{2} = \frac{(3.9\text{ mm} - 3.1\text{ mm})\left(\dfrac{1\text{ m}}{1000\text{ mm}}\right)}{2} = \underline{\hspace{3cm}}\text{ m}$$

Substituting this parameter into the definition of Biot number, we get:

$$N_{Bi} = \frac{\left(12\dfrac{\text{W}}{\text{m}^2\cdot\text{K}}\right)(\underline{\hspace{3cm}}\text{ m})}{\left(2.53\dfrac{\text{W}}{\text{m}\cdot\text{K}}\right)} = \underline{\hspace{3cm}}$$

Since this value is less than 0.1, we can proceed to enter the corresponding values into the equation for the dimensionless temperature, to yield:

$$\frac{340.23\text{ K} - \underline{\hspace{2cm}}\text{ K}}{\underline{\hspace{2cm}}\text{ K} - \underline{\hspace{2cm}}\text{ K}} = e^{-\left[\frac{12\frac{\text{W}}{\text{m}^2\cdot\text{K}}}{\left(594.3\frac{\text{J}}{\text{kg}\cdot\text{K}}\right)\left(6337.3\frac{\text{kg}}{\text{m}^3}\right)(\underline{\hspace{2cm}}\text{ m})}\right]t}$$

Solving for the time, we have:

$$t = -\frac{1}{(\underline{\hspace{3cm}}\text{ s}^{-1})}\ln\left(\frac{340.23\text{ K} - \underline{\hspace{2cm}}\text{ K}}{\underline{\hspace{2cm}}\text{ K} - \underline{\hspace{2cm}}\text{ K}}\right)$$

$$t = \underline{\hspace{2cm}}\text{ s}\left(\frac{\underline{\hspace{2cm}}}{\underline{\hspace{2cm}}}\right)$$

$$\boxed{t = \underline{\hspace{2cm}}\text{ min}}$$

Example 5.2-2: Total Amount of Heat in Cooling of a Solid – Oxide Fuel Cell

Determine the total amount of heat removed from the fuel cell in Problem 5.2-1 after 5 minutes. The characteristic length of the fuel cell is $4\times10^{-4}\,\text{m}$.

Strategy

The equation for the total amount of heat in J can be used to solve this problem.

Solution

The equation for the total amount of heat is given by:

$$Q = C_p\rho V(T_0 - T_\infty)\left[1 - e^{-\left(\frac{h}{C_p\rho x_1}\right)t}\right]$$

The volume of the cylindrical fuel cell is calculated as follows:

$$V = \pi(\_\_\_\_\_\_ - \_\_\_\_\_\_)L$$

$$V = \pi[\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ - \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_]0.2\text{ m}$$

$$V = 3.52\times10^{-6}\,\text{m}^3$$

Substituting the calculated volume and the rest of the values into the equation for the total amount of heat transferred, yields:

$$Q = \left(594.3\frac{\text{J}}{\text{kg}\cdot\text{K}}\right)\left(6337.3\frac{\text{kg}}{\text{m}^3}\right)\left(\_\_\_\_\_\_\_\_\_\_\ \text{m}^3\right)(873.15\text{ K} - 333.15\text{ K})\left[1 - e^{-\left(\frac{12\frac{\text{W}}{\text{m}^2\cdot\text{K}}}{594.3\frac{\text{J}}{\text{kg}\cdot\text{K}}\left(6337.3\frac{\text{kg}}{\text{m}^3}\right)\left(4\times10^{-4}\,\text{m}\right)}\right)\_\_\_\_\ \text{s}}\right]$$

$$\boxed{Q = \_\_\_\_\_\_\_\_\_\_\_\_\ \text{J}}$$

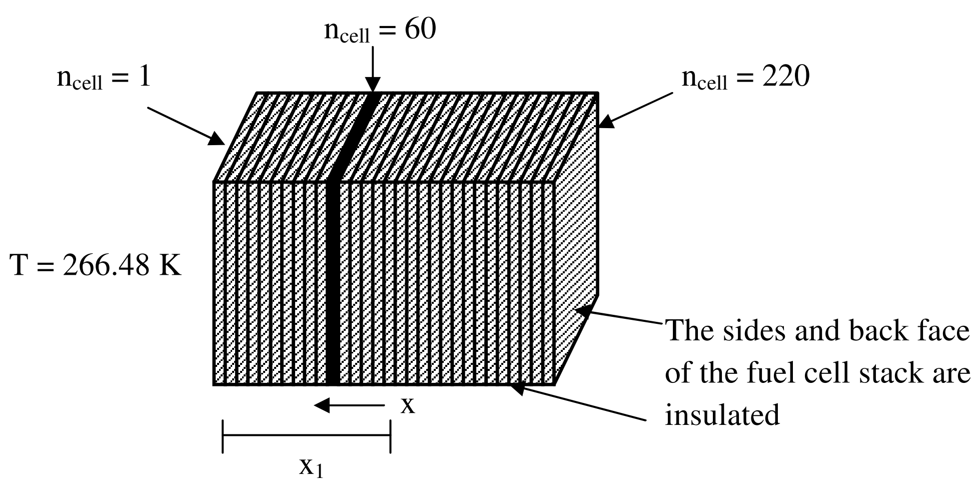
Example 5.3-2: Heat Conduction in a Fuel Cell Stack

A stack of 220 fuel cells is initially at a temperature of 353.15 K. This stack enters a room at a temperature of 266.48 K. Assuming all the sides but the front face of the stack are insulated, determine the temperature of the 60^{th} fuel cell after 90 minutes. The thickness of the bipolar plates in this fuel cell stack is 2 mm and it is much thicker than the membrane electrode assembly. The heat transfer coefficient of the air in this room is $91 \frac{W}{m^2 \cdot K}$

The properties of the bipolar plates in this fuel cell stack are given below [2]:

$$\rho = 1632 \frac{kg}{m^3} \qquad C_p = 1414 \frac{J}{kg \cdot K} \qquad k = 24.42 \frac{W}{m \cdot K}$$

A schematic of this fuel cell stack is shown in the following figure:



Strategy

The temperature at a point inside the fuel cell stack can be obtained from the charts for unsteady – state heat conduction of a slab.

Solution

To use Figure 5.3-5 of Geankoplis to determine the unsteady – state heat conduction in a flat plate, we need to find the parameters X, m and n, defined as:

$$X = \_\_\_\_\_\_\_\_ \qquad m = \frac{k}{hx_1} \qquad n = \_\_\_\_\_\_\_\_\_\_$$

2. King, J.A., Lopez Gaxiola, D., Johnson, B.A., Keith, J.M., *Journal of Composite Materials*, 44 (7), 839 – 855 (2010).

The characteristic length x_1 in these equations corresponds to the distance from the front face to the center of the fuel cell stack, obtained by multiplying the number of fuel cells by the thickness of a bipolar plate, and dividing it by 2. Thus,

$$x_1 = \frac{\_\_\_\_\_\_\_\ \text{m}(220)}{2}$$

$$x_1 = \_\_\_\_\_\_\_\_\_\ \text{m}$$

The parameter α is the diffusivity of the fuel cell stack defined as:

$$\alpha = \frac{k}{\_\_\_\_\_\_}$$

Substituting the properties of the bipolar plates into this equation, we get:

$$\alpha = \frac{24.42\,\frac{\text{W}}{\text{m}\cdot\text{K}}}{\_\_\_\_\_\_\_\_\ \frac{\text{kg}}{\text{m}^3}\left(\_\_\_\_\_\_\_\_\ \frac{\text{J}}{\text{kg}\cdot\text{K}}\right)} = \_\_\_\_\_\_\_\_\_\_\_\_\ \frac{\text{m}^2}{\text{s}}$$

To determine the value of n, we need to know the distance at which the 60^{th} fuel cell is located from the center of the stack. This distance can be calculated by multiplying the thickness of a single bipolar plate by (110-60) which is the number of fuel cells from the center.

$$x = (110-60)(\_\_\_\_\_\_\_\_\ \text{m}) = \_\_\_\_\_\_\_\_\ \text{m}$$

Now we can calculate the three values required to use Figure 5.3-5, as shown in the following steps:

$$X = \frac{\left(\_\_\_\_\_\_\_\_\_\_\_\_\ \frac{\text{m}^2}{\text{s}}\right)(90\text{ min})\left(\frac{60\text{ s}}{1\text{ min}}\right)}{(\_\_\_\_\_\_\_\_\ \text{m})^2} = \_\_\_\_\_\_\_\_$$

$$m = \frac{24.42\,\frac{\text{W}}{\text{m}\cdot\text{K}}}{91\,\frac{\text{W}}{\text{m}^2\cdot\text{K}}(\_\_\_\_\_\_\_\_\ \text{m})} = 1.22$$

$$n = \frac{\_\_\_\_\_\_\_\_\ \text{m}}{\_\_\_\_\_\_\_\_\ \text{m}} = 0.45$$

From Figure 5.3-5 we can read a value for a dimensionless temperature Y = _______. This value can be used to solve for the temperature in the 60^{th} fuel cell as follows:

$$Y = \frac{T_1 - T}{T_1 - T_0} = \_\_\_\_\_\_\_\_\_$$

where:

T_1 = Temperature of the cooling medium

T_0 = Initial temperature of the fuel cell stack.

We can enter the corresponding temperatures and solve for the temperature T to get:

$$T = T_1 - \_\_\_\_\_\_\_\_\_\left(\_\_\_\_\_\_\_ - \_\_\_\_\_\_\_\right)$$

$$T = 266.48\text{ K} - \_\_\_\_\_\_\_\_\_\left(266.48\text{ K} - \_\_\_\_\_\_\_\_\_\text{ K}\right)$$

$$\boxed{T = \_\_\_\_\_\_\_\_\_\text{ K}}$$

Example 5.3-3: Transient Heat Conduction in a Cylindrical Solid – Oxide Fuel Cell

A cylindrical solid – oxide fuel cell with a diameter of 3.9 mm and a length of 0.2 m is initially at a temperature of 1150 K. The fuel cell is shut down in a room where the air is at a temperature of 303 K. Calculate the temperature at the center of the fuel cell after 5 minutes, assuming it is insulated on the flat ends. The heat transfer coefficient of the air is $10 \frac{\text{W}}{\text{m}^2 \cdot \text{K}}$. The thermal conductivity and diffusivity of the fuel cell are $2.53 \frac{\text{W}}{\text{m} \cdot \text{K}}$ and $6.72 \times 10^{-7} \frac{\text{m}^2}{\text{s}}$, respectively.

Strategy

The solution to this problem can be found by using the charts for unsteady – state heat conduction in a cylinder.

Solution

Since heat is only being transferred through the walls, the fuel cell can be considered as a long cylinder. The parameters n, m and X required to determine the dimensionless temperature Y from Figure 5.3-8 are calculated as shown in the following steps:

At the center of the cylinder, x = 0. Thus, the value of n will also be equal to zero:

$$n = \frac{x}{x_1} = \frac{0 \text{ m}}{\underline{\hspace{3cm}} \text{ m}}$$

$$n = 0$$

To calculate the value of X, we need to substitute the diffusivity, radius of the fuel cell and the time elapsed, as shown below:

$$X = \frac{\alpha t}{x_1^2}$$

$$X = \frac{\left(\underline{\hspace{3cm}} \frac{\text{m}^2}{\text{s}}\right)\left(\underline{\hspace{1cm}} \text{ min}\right)\left(\frac{60 \text{ s}}{1 \text{ min}}\right)}{\left(\underline{\hspace{3cm}} \text{ m}\right)^2} = \underline{\hspace{2cm}}$$

Finally, we can determine the parameter m:

$$m = \frac{k}{h x_1}$$

$$m = \frac{\underline{\hspace{2cm}}\ \frac{W}{m \cdot K}}{\underline{\hspace{2cm}}\ \frac{W}{m^2 \cdot K}\left(3.9\times10^{-3}\ m\right)} = \underline{\hspace{2cm}}$$

For these values, the corresponding Y from Figure 5.3-8 will be equal to ______. Now we can solve for the temperature at the center of the fuel cell from the definition of the dimensionless temperature Y:

$$Y = \frac{T_1 - T}{\underline{\hspace{2.5cm}}} = \underline{\hspace{1.5cm}}$$

$$T = T_1 - \underline{\hspace{5cm}}$$

where:

T_1 = Temperature of cooling medium

T_0 = Initial temperature of the fuel cell

Substituting the temperature values in this equation, we get:

$$T = 303\ K - \underline{\hspace{1.5cm}}\left(303\ K - \underline{\hspace{2cm}}\ K\right)$$

$$\boxed{T = \underline{\hspace{2.5cm}}\ K}$$

Example 5.3-4: Two – Dimensional Conduction in a Cylindrical Solid – Oxide Fuel Cell

Determine the temperature at the center of the solid – oxide fuel cell from Example 5.3-3 now considering heat conduction also occurring through the ends of the cylinder. What does this result indicate?

Strategy

The procedure to solve this problem consists of using the charts for unsteady – state heat transfer in a cylinder for both radial and axial directions.

Solution

We need to calculate the required dimensionless quantities X, m and n for both radial and axial directions. For the radial direction, these will be given by:

$$n_{radial} = \frac{x}{x_1} = \frac{\_\_\_\_\_\_ \text{ m}}{\_\_\_\_\_\_\_\_\_\_\_\_\_ \text{ m}}$$

$$n_{radial} = \_\_\_\_$$

$$m_{radial} = \frac{k}{hx_1}$$

$$m_{radial} = \frac{\_\_\_\_\_\_\_\_ \frac{\text{W}}{\text{m} \cdot \text{K}}}{\_\_\_\_ \frac{\text{W}}{\text{m}^2 \cdot \text{K}} \left(3.9 \times 10^{-3} \text{ m}\right)} = \_\_\_\_\_\_\_$$

$$X_{radial} = \frac{\alpha t}{x_1^2}$$

$$X_{radial} = \frac{\left(\_\_\_\_\_\_\_\_\_\_\_\_\_\_ \frac{\text{m}^2}{\text{s}}\right)(5 \text{ min})\left(\frac{60 \text{ s}}{1 \text{ min}}\right)}{(\_\_\_\_\_\_\_\_\_\_\_\_\_ \text{ m})^2} = \_\_\_\_\_\_\_\_$$

These values will yield a dimensionless temperature Y_{radial} = ______

For heat conduction in the axial direction, we need to calculate the parameters n, m and X and locate them in Figure 5.3-6, applicable for two parallel planes. Thus,

$$n_{axial} = \frac{y}{y_1} = \frac{0 \text{ m}}{0.1 \text{ m}}$$

$$n_{axial} = 0$$

$$m_{axial} = \frac{k}{hy_1}$$

$$m_{axial} = \frac{\_\_\_\_\_\_ \frac{W}{m \cdot K}}{\_\_\_\_\_\_ \frac{W}{m^2 \cdot K}(\_\_\_\_\_\_\ m)} = \_\_\_\_\_\_$$

$$X_{axial} = \frac{\alpha t}{x_1^2}$$

$$X_{axial} = \frac{\left(\_\_\_\_\_\_\_\_\_\_\_\_ \frac{m^2}{s}\right)(\_\_\_\_\_\_\ min)(\_\_\_\_\_\_)}{(\_\_\_\_\_\_\ m)^2} = \_\_\_\_\_\_$$

Locating these three parameters in Figure 5.3-6 gives a value of Y_{axial} = ___.

Now that we have both Y values for both directions, we can obtain a Y for the overall heat transfer process as follows:

$$Y = Y_{axial} Y_{radial} = \_\_\_\_\_\_\_\_$$

Now we can solve for the temperature at the center of the cylinder to get:

$$Y = \frac{T_1 - T}{T_1 - T_0} = \_\_\_\_\_\_\_\_\_\_$$

$$T = T_1 - \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

$$T = \_\_\_\_\_\_\ K - \_\_\_\_\_\_(\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_)$$

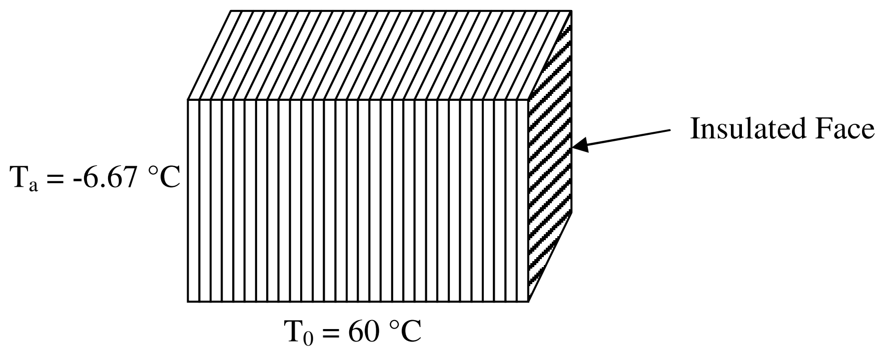
$$\boxed{T = \_\_\_\_\_\_\_\_\_\ K}$$

Conclusion:

Example 5.4-1: Unsteady – State Conduction and the Schmidt Numerical Method

A proton – exchange membrane fuel cell stack has a thickness of 0.3 m and is initially at a uniform temperature of 60°C. The front face of the fuel cell stack suddenly exposed to an environmental temperature of -6.67°C. The bulk thermal diffusivity of the fuel cell stack is $8.69\times10^{-6}\,\frac{m^2}{s}$. Assuming the convective resistance is negligible and that the back face of the stack is insulated, determine the temperature profile after 24 minutes using the Schmidt numerical method with M = 2 and dividing the fuel cell stack into slices with a thickness of 0.05 m. Follow the special procedure for the first time increment.

The next figure illustrates the conditions in this cooling process:



Strategy

The equations we need to use to determine the temperature profile will depend on the boundary conditions.

Solution

The number of time steps to use in this problem will be determined from the definition of the parameter M:

$$M = \frac{(\Delta x)^2}{\alpha \Delta t}$$

Solving for the time increment Δt and substituting the corresponding values into this equation, yields:

$$\Delta t = \frac{(\Delta x)^2}{M\alpha} = \frac{(\_\_\_\_\_\_\ m)^2}{2\left(\_\_\_\_\_\_\_\_\_\_\_\_\ \frac{m^2}{s}\right)}$$

$\Delta t =$ _________ s

The number of time steps needed for this Δt is given by:

$$n_{\text{time steps}} = \frac{t}{\Delta t} = \frac{(\_\_\_\_\_\_\ \text{min})(\_\_\_\_\_\_)}{\_\_\_\_\_\_\_\_\ \text{s}} \approx 10$$

The front surface of the fuel cell stack corresponds to n =1. In this point, the temperature ${}_1T_a$ used for the first time increment is given by:

$${}_1T_a = \frac{T_a + {}_0T_1}{2}$$

where:

${}_0T_1$ = Initial temperature at point 1.

T_a = Temperature of the environment = -6.67°C

Since there is no convective heat resistance at the interface, for the remaining time increments:

$$T_1 = T_a$$

The general equation for determining the temperature for the slabs n = 2 to 6 is given below:

$${}_{t+\Delta t}T_n = \frac{{}_tT_{n-1} + {}_tT_{n+1}}{2}$$

We need an additional equation for the insulated face. This is the point where n = 7 and its corresponding equation is given by:

$${}_{t+\Delta t}T_7 = \frac{(M-2)\,{}_tT_7 + 2\,{}_tT_6}{M}$$

Now we can proceed to calculate the temperatures for the first time increment. Thus, for n = 1,

$${}_{t+\Delta t}T_1 = \frac{T_a + {}_0T_1}{2} = \frac{-6.67°C + \_\_\_\_\_\_°C}{2} = \_\_\_\_\_\_\ °C$$

Since the problem indicates we should use the special procedure for the first time increment, this temperature value we just obtained is equal to the ambient temperature at the first time increment:

$${}_{t+\Delta t}T_1 = {}_1T_a$$

For n = 2:

$$_{t+\Delta t}T_2 = \frac{_tT_1 + _tT_3}{2}$$

The special procedure used at n = 1 for the first time increment also affects this equation: instead of using the temperature of -6.67°C we will use $_1T_a$. Hence,

$$_{t+\Delta t}T_2 = \frac{_1T_a + _tT_3}{2} = \frac{\_\_\_\_\_\_\_\_ °C + \_\_\_\_\_\_ °C}{2} = \_\_\_\_\_\_\_\_ °C$$

The following four slabs corresponding to n = 3 to 6 can be calculated with the general equation for n = 2 to 6, given in previous steps. Substituting the corresponding temperatures into this equation yields:

$$_{t+\Delta t}T_3 = \frac{_tT_2 + _tT_4}{2} = \frac{60°C + 60°C}{2} = 60°C$$

$$_{t+\Delta t}T_4 = \frac{_tT_3 + _tT_5}{2} = \frac{60°C + 60°C}{2} = 60°C$$

$$_{t+\Delta t}T_5 = \frac{_tT_4 + _tT_6}{2} = \frac{60°C + 60°C}{2} = 60°C$$

$$_{t+\Delta t}T_6 = \frac{_tT_5 + _tT_7}{2} = \frac{60°C + 60°C}{2} = 60°C$$

For n = 7, we use the equation for the insulated face:

$$_{t+\Delta t}T_7 = \frac{(2-2)_tT_7 + 2_tT_6}{2} = _tT_6 = 60°C$$

Now we can proceed to calculate the temperatures from the second to the tenth time increments, as shown in the following steps:

For 2Δt:

$$_{t+2\Delta t}T_1 = T_a = \_\_\_\_\_\_ °C$$

$$_{t+2\Delta t}T_2 = \frac{_{t+\Delta t}T_1 + _{t+\Delta t}T_3}{2} = \frac{\_\_\_\_\_\_ + 60°C}{2} = \_\_\_\_\_\_ °C$$

$$_{t+2\Delta t}T_3 = \frac{_{t+\Delta t}T_2 + _{t+\Delta t}T_4}{2} = \frac{\_\_\_\_\_\_ °C + 60°C}{2} = \_\_\_\_\_\_ °C$$

$$_{t+2\Delta t}T_4 = \frac{_{t+\Delta t}T_3 + _{t+\Delta t}T_5}{2} = \frac{\_\_\_\_\_\_ °C + 60°C}{2} = \_\_\_\_\_\_ °C$$

$$ {}_{t+2\Delta t}T_5 = \frac{{}_{t+\Delta t}T_4 + {}_{t+\Delta t}T_6}{2} = \frac{\_\_\_\_\_\,°C + 60°C}{2} = \_\_\_\_\_\_\,°C $$

$$ {}_{t+2\Delta t}T_6 = \frac{{}_{t+\Delta t}T_5 + {}_{t+\Delta t}T_7}{2} = \frac{\_\_\_\_\_\,°C + 60°C}{2} = \_\_\_\_\_\_\,°C $$

$$ {}_{t+2\Delta t}T_7 = {}_{t+\Delta t}T_6 = \_\_\_\_\_\_\,°C $$

For 3Δt:

$$ {}_{t+3\Delta t}T_1 = T_a = -6.67°C $$

$$ {}_{t+3\Delta t}T_2 = \frac{{}_{t+2\Delta t}T_1 + {}_{t+2\Delta t}T_3}{2} = \frac{-6.67°C + \_\_\_\_\_\,°C}{2} = \_\_\_\_\_\_\,°C $$

$$ {}_{t+3\Delta t}T_3 = \frac{{}_{t+2\Delta t}T_2 + {}_{t+2\Delta t}T_4}{2} = \frac{\_\_\_\_\_\,°C + \_\_\_\_\_\,°C}{2} = \_\_\_\_\_\_\,°C $$

$$ {}_{t+3\Delta t}T_4 = \frac{{}_{t+2\Delta t}T_3 + {}_{t+2\Delta t}T_5}{2} = \frac{\_\_\_\_\_\,°C + \_\_\_\_\_\,°C}{2} = \_\_\_\_\_\_\,°C $$

$$ {}_{t+3\Delta t}T_5 = \frac{{}_{t+2\Delta t}T_4 + {}_{t+2\Delta t}T_6}{2} = \frac{\_\_\_\_\_\,°C + \_\_\_\_\_\,°C}{2} = \_\_\_\_\_\_\,°C $$

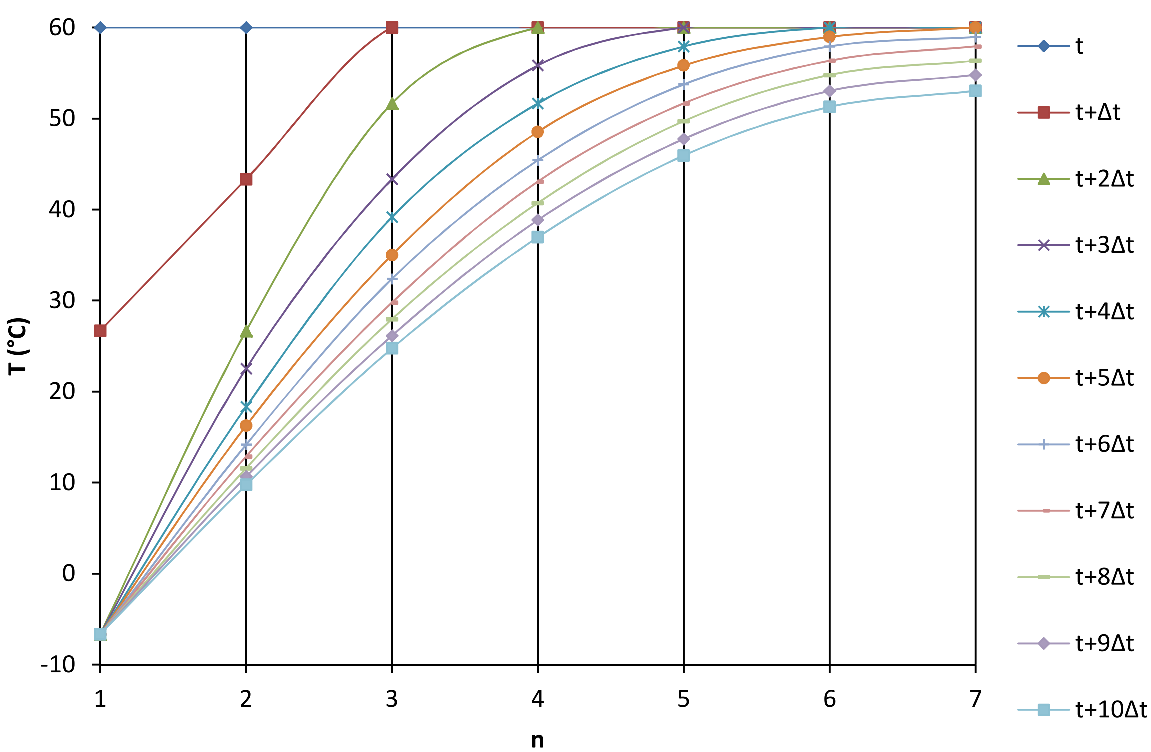
$$ {}_{t+3\Delta t}T_6 = \frac{{}_{t+2\Delta t}T_5 + {}_{t+2\Delta t}T_7}{2} = \frac{\_\_\_\_\_\,°C + \_\_\_\_\_\,°C}{2} = \_\_\_\_\_\_\,°C $$

$$ {}_{t+3\Delta t}T_7 = {}_{t+2\Delta t}T_6 = \_\_\_\_\,°C $$

We can continue to repeat this procedure up to 10ΔT. The solutions for the all the temperatures (in °C) are given in the following table:

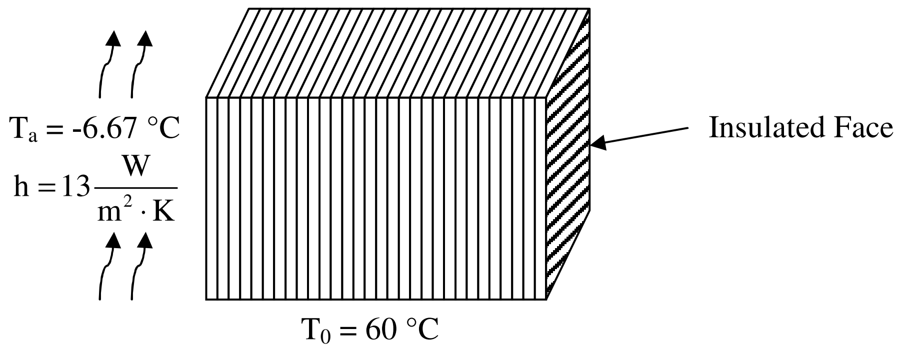
	n						
	1	2	3	4	5	6	7
t	60	60	60	______	60	______	60
t+Δt	______	______	60	60	______	60	______
t+2Δt	-6.67	______	______	60	______	______	______
t+3Δt	______	22.50	______	______	60	______	60
t+4Δt	-6.67	______	39.17	______	______	60	______
t+5Δt	-6.67	______	______	48.54	______	______	______
t+6Δt	-6.67	14.16	______	______	53.75	______	58.96
t+7Δt	-6.67	______	29.79	______	______	56.35	______
t+8Δt	-6.67	______	______	40.73	______	______	______
t+9Δt	-6.67	10.65	______	______	47.76	______	54.79
t+10Δt	-6.67	______	24.74	______	______	51.28	______

We can plot these temperature results to observe the change in temperature within the fuel cell stack:



Example 5.4-3: Unsteady – State Conduction with Convective Boundary Condition

Determine the temperature profile for the same fuel cell stack as in Example 5.4-1, now with a convective coefficient of $13 \dfrac{\mathrm{W}}{\mathrm{m^2 \cdot K}}$. The bulk thermal conductivity of the fuel cell stack is $20 \dfrac{\mathrm{W}}{\mathrm{m \cdot K}}$. Use a value of M = 4.



Strategy

In this problem we need to use Schmidt numerical method depending on the boundary conditions.

Solution

Before we start applying Schmidt method, we need to determine the number of time increments we need to use. From the definition of the parameter M, we can solve for the time increment Δt, as shown in the following steps:

$$M = \frac{(\Delta x)^2}{\alpha \Delta t}$$

$$\Delta t = \frac{(\Delta x)^2}{\alpha M} = \frac{(0.05 \text{ m})^2}{\left(8.69 \times 10^{-6} \dfrac{\mathrm{m^2}}{\mathrm{s}}\right)(4)} = \underline{\qquad\qquad} \text{ s}$$

The number of time steps needed for this Δt is given by:

$$n_{\text{time steps}} = \frac{t}{\Delta t} = \frac{(\underline{\qquad} \text{ min})\left(\dfrac{60 \text{ s}}{1 \text{ min}}\right)}{71.92 \text{ s}} \approx \underline{\qquad\quad}$$

As stated in Section 5.4B of Geankoplis, when the value of M is greater than 3, the value of the environmental temperature T_a will be the same for all time increments. Therefore,

$$T_a = -6.67°C$$

For the node n = 1, corresponding to the front face of the fuel cell stack, the temperature can be calculated using Equation 5.4-7 of Geankoplis:

$$_{t+\Delta t}T_1 = \frac{1}{M}\left\{2N_t T_a + \left[M-(2N+2)\right]_t T_1 + 2_t T_2\right\}$$

The value of N is a function of the convective heat transfer coefficient and the thermal conductivity of the fuel cell, described by the following equation:

$$N = \frac{h\Delta x}{k}$$

Entering the corresponding values into this equation, yields:

$$N = \frac{\underline{\qquad}\frac{W}{m^2 \cdot K}\left(\underline{\qquad}\ m\right)}{20\frac{W}{m \cdot K}} = \underline{\qquad\qquad}$$

In order to use Equation 5.4-7, the value of M must satisfy the following constraint:

$$M \geq 2N+2$$

Substituting numeric quantities into this constraint, we get:

$$4 \geq 2(0.0325)+2$$

$$4 \geq \underline{\qquad}$$

Hence, we can use Equation 5.4-7. From this equation, we can find the temperature $_{t+\Delta t}T_1$ to be:

$$_{t+\Delta t}T_1 = \frac{1}{4}\left\{2\left(\underline{\qquad\quad}\right)\left(-6.67°C\right)+\left[4-\left(\underline{\qquad\quad}\right)\right]\left(60°C\right)+2(60°C)\right\}$$

$$_{t+\Delta t}T_1 = \underline{\qquad\quad}°C$$

For the points at n = 2, 3, 4, 5, 6, we use Equation 5.4-2:

$$_{t+\Delta t}T_n = \frac{1}{M}\left[{}_tT_{n+1} + \left(M-2\right)_t T_n + {}_t T_{n-1}\right]$$

By entering the value of M into this equation, we can get a general equation for $_{t+\Delta t}T_n$:

$$_{t+\Delta t}T_n = \underline{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}$$

Now we need an equation for the insulated boundary at n = 7, which is Equation 5.4-10 of Geankoplis:

$$_{t+\Delta t}T_7 = \frac{(M-2)_t T_7 + 2_t T_6}{M}$$

$$_{t+\Delta t}T_7 = \frac{_t T_7 +_t T_6}{\_\_\_\_\_}$$

We can now proceed to determine the temperatures for n = 2 to 7 for the first time increment. Thus,

$$_{t+\Delta t}T_2 = 0.25(_t T_3 +_t T_1) + 0.5_t T_2$$

$$_{t+\Delta t}T_2 = 0.25(\_\_\_\_\_\_ °C + 60°C) + 0.5(\_\_\_\_\_\_ °C) = \_\_\_\_\_\_ °C$$

$$_{t+\Delta t}T_3 = 0.25(_t T_4 +_t T_2) + 0.5_t T_3$$

$$_{t+\Delta t}T_3 = 0.25(\_\_\_\_\_\_ °C + 60°C) + 0.5(\_\_\_\_\_\_ °C) = \_\_\_\_\_\_ °C$$

$$_{t+\Delta t}T_4 = 0.25(_t T_5 +_t T_3) + 0.5_t T_4$$

$$_{t+\Delta t}T_4 = 0.25(\_\_\_\_\_\_ °C + 60°C) + 0.5(\_\_\_\_\_\_ °C) = \_\_\_\_\_\_ °C$$

$$_{t+\Delta t}T_5 = 0.25(_t T_6 +_t T_4) + 0.5_t T_5$$

$$_{t+\Delta t}T_5 = 0.25(\_\_\_\_\_\_ °C + 60°C) + 0.5(\_\_\_\_\_\_ °C) = \_\_\_\_\_\_ °C$$

$$_{t+\Delta t}T_6 = 0.25(_t T_7 +_t T_5) + 0.5_t T_6$$

$$_{t+\Delta t}T_6 = 0.25(\_\_\_\_\_\_ °C + 60°C) + 0.5(\_\_\_\_\_\_ °C) = \_\_\_\_\_\_ °C$$

$$_{t+\Delta t}T_7 = \frac{_t T_7 +_t T_6}{2}$$

$$_{t+\Delta t}T_7 = \frac{(\_\_\_\_ °C) + (\_\_\_\_ °C)}{2} = \_\_\_\_ °C$$

For 2Δt:

$$_{t+2\Delta t}T_1 = \frac{1}{4}\left[0.065_{t+\Delta t}T_a + \_\_\_\_\_\_\_\_ {}_{t+\Delta t}T_1 + 2_{t+\Delta t}T_2\right]$$

$$_{t+2\Delta t}T_1 = \frac{1}{4}\left[0.065_{t+\Delta t}(\_\_\_\_\_\_\ °C) + \_\_\_\_\_\_{}_{t+\Delta t}(\_\_\_\_\_\_\ °C) + 2_{t+\Delta t}(60°C)\right] = \_\_\_\_\_\_\ °C$$

$$_{t+2\Delta t}T_2 = 0.25\left({}_{t+\Delta t}T_3 + {}_{t+\Delta t}T_1\right) + 0.5_{t+\Delta t}T_2$$

$$_{t+2\Delta t}T_2 = 0.25\left(\_\_\_\_\ °C + 58.92°C\right) + 0.5(60°C) = 59.73°C$$

$$_{t+2\Delta t}T_3 = 0.25\left({}_{t+\Delta t}T_4 + {}_{t+\Delta t}T_2\right) + 0.5_{t+\Delta t}T_3$$

$$_{t+2\Delta t}T_3 = 0.25\left(\_\_\_\_\ °C + 60°C\right) + 0.5(\_\_\_\_\ °C) = \_\_\_\_\ °C$$

$$_{t+2\Delta t}T_4 = 0.25\left({}_{t+\Delta t}T_5 + {}_{t+\Delta t}T_3\right) + 0.5_{t+\Delta t}T_4$$

$$_{t+2\Delta t}T_4 = 0.25\left(\_\_\_\_\ °C + \_\_\_\_\ °C\right) + 0.5(\_\_\_\_\ °C) = \_\_\_\_\ °C$$

$$_{t+2\Delta t}T_5 = 0.25\left({}_{t+\Delta t}T_6 + {}_{t+\Delta t}T_4\right) + 0.5_{t+\Delta t}T_5$$

$$_{t+2\Delta t}T_5 = 0.25\left(\_\_\_\_\ °C + \_\_\_\_\ °C\right) + 0.5(\_\_\_\_\ °C) = \_\_\_\_\ °C$$

$$_{t+2\Delta t}T_6 = 0.25\left({}_{t+\Delta t}T_7 + {}_{t+\Delta t}T_5\right) + 0.5_{t+\Delta t}T_6$$

$$_{t+2\Delta t}T_6 = 0.25\left(\_\_\_\_\ °C + \_\_\_\_\ °C\right) + 0.5(\_\_\_\_\ °C) = \_\_\_\_\ °C$$

$$_{t+2\Delta t}T_7 = \frac{{}_{t+\Delta t}T_7 + {}_{t+\Delta t}T_6}{2}$$

$$_{t+2\Delta t}T_7 = \frac{(\_\_\_\_\ °C) + (\_\_\_\_\ °C)}{2} = \_\_\_\_\ °C$$

For $3\Delta t$:

$$_{t+3\Delta t}T_1 = \frac{1}{4}\left[0.065_{t+2\Delta t}T_a + 1.935_{t+2\Delta t}T_1 + 2_{t+2\Delta t}T_2\right]$$

$$_{t+3\Delta t}T_1 = \frac{1}{4}\left[0.065_{t+2\Delta t}(\_\_\_\_\_\ °C) + 1.935_{t+2\Delta t}(\_\_\_\_\_\_\ °C) + 2_{t+2\Delta t}(59.73°C)\right] = \_\_\_\_\_\ °C$$

$$_{t+3\Delta t}T_2 = 0.25\left({}_{t+2\Delta t}T_3 + {}_{t+2\Delta t}T_1\right) + 0.5_{t+2\Delta t}T_2$$

$$_{t+3\Delta t}T_2 = 0.25\left(60°C + \_\_\_\_\_\ °C\right) + 0.5(59.73°C) = \_\_\_\_\_\ °C$$

$$_{t+3\Delta t}T_3 = 0.25\left({}_{t+2\Delta t}T_4 + {}_{t+2\Delta t}T_2\right) + 0.5_{t+2\Delta t}T_3$$

$$_{t+3\Delta t}T_3 = 0.25(\_\_\_\_\_\_\_\,°C + \_\_\_\_\_\_\_\,°C) + 0.5(\_\_\_\_\_\_\_\,°C) = \_\_\_\_\_\_\_\,°C$$

$$_{t+3\Delta t}T_4 = 0.25({}_{t+2\Delta t}T_5 + {}_{t+2\Delta t}T_3) + 0.5\,{}_{t+2\Delta t}T_4$$

$$_{t+3\Delta t}T_4 = 0.25(\_\_\_\_\_\_\_\,°C + \_\_\_\_\_\_\_\,°C) + 0.5(\_\_\_\_\_\_\_\,°C) = \_\_\_\_\_\_\_\,°C$$

$$_{t+3\Delta t}T_5 = 0.25({}_{t+2\Delta t}T_6 + {}_{t+2\Delta t}T_4) + 0.5\,{}_{t+2\Delta t}T_5$$

$$_{t+3\Delta t}T_5 = 0.25(\_\_\_\_\_\_\_\,°C + \_\_\_\_\_\_\_\,°C) + 0.5(\_\_\_\_\_\_\_\,°C) = \_\_\_\_\_\_\_\,°C$$

$$_{t+3\Delta t}T_6 = 0.25({}_{t+2\Delta t}T_7 + {}_{t+2\Delta t}T_5) + 0.5\,{}_{t+2\Delta t}T_6$$

$$_{t+3\Delta t}T_6 = 0.25(\_\_\_\_\_\_\_\,°C + \_\_\_\_\_\_\_\,°C) + 0.5(\_\_\_\_\_\_\_\,°C) = \_\_\_\_\_\_\_\,°C$$

$$_{t+3\Delta t}T_7 = \frac{{}_{t+2\Delta t}T_7 + {}_{t+2\Delta t}T_6}{2}$$

$$_{t+3\Delta t}T_7 = \frac{(\_\_\_\_\,°C) + (\_\_\_\_\,°C)}{2} = \_\_\_\_\,°C$$

In a similar way, we can continue to perform this calculation for all the 20 time increments. The results for all temperatures (in °C) are shown in the following table:

	n						
	1	2	3	4	5	6	7
t	60	60	60	60	60	60	60
t+Δt	_______	_______	_______	60	_______	_______	60
t+2Δt	_______	59.73	_______	_______	60	_______	_______
t+3Δt	58.00	_______	59.93	_______	_______	60	_______
t+4Δt	_______	_______	_______	59.98	_______	_______	60
t+5Δt	_______	58.99	_______	_______	60	_______	_______
t+6Δt	57.15	_______	59.59	_______	_______	60	_______
t+7Δt	_______	_______	_______	59.85	_______	_______	60
t+8Δt	_______	58.38	_______	_______	59.94	_______	_______
t+9Δt	56.52	_______	59.21	_______	_______	59.98	_______
t+10Δt	_______	_______	_______	59.64	_______	_______	59.99
t+11Δt	_______	57.87	_______	_______	59.84	_______	_______
t+12Δt	56.00	_______	58.84	_______	_______	59.93	_______
t+13Δt	_______	_______	_______	59.40	_______	_______	59.95
t+14Δt	_______	57.42	_______	_______	59.70	_______	_______
t+15Δt	55.54	_______	58.48	_______	_______	59.85	_______
t+16Δt	_______	_______	_______	59.15	_______	_______	59.87
t+17Δt	_______	57.02	_______	_______	59.54	_______	_______
t+18Δt	55.14	_______	58.15	_______	_______	59.73	_______
t+19Δt	_______	_______	_______	58.90	_______	_______	59.77
t+20Δt	_______	56.65	_______	_______	59.36	_______	_______

The graphical representation of these temperatures is shown in the following figure: